Revision 4 (Solutions)

Year 11 Examination

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2

Section Two: Calculator-assumed

Student Number: In figures | | | | | | | | |

In words ______________________________

Teacher name ______________________________

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

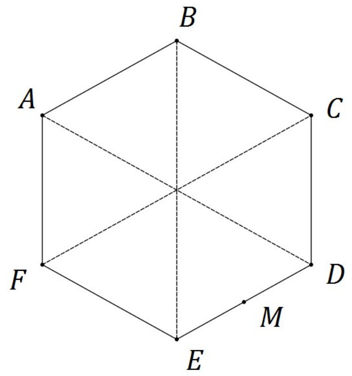
Section Two: Calculator-assumed **65% (58 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 60 minutes.

Question 1 **(7 marks)**

(a) $ABCDEF$ is a regular hexagon. The midpoint of side DE is M.



Let $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{AF}$. Express each of the following in terms of $\mathbf{a}$ and $\mathbf{b}$.

(i) $\overrightarrow{BC}$. (1 mark)

Solution
$\overrightarrow{BC} = \mathbf{a} + \mathbf{b}$
Specific behaviours
✓ states correct expression

(ii) $\overrightarrow{AE}$. (1 mark)

Solution
$\overrightarrow{AE} = \overrightarrow{AF} + \overrightarrow{FE} = \mathbf{b} + (\mathbf{a} + \mathbf{b}) = \mathbf{a} + 2\mathbf{b}$
Specific behaviours
✓ states correct expression

(iii) $\overrightarrow{MB}$. (1 mark)

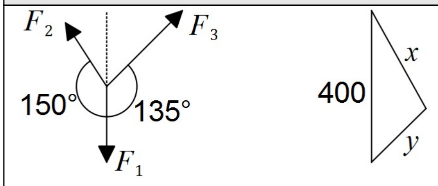
Solution
$\overrightarrow{MB} = \overrightarrow{ME} + \overrightarrow{BE} = -0.5\mathbf{a} - 2\mathbf{b}$
Specific behaviours
✓ states correct expression

(b) Three forces, F_1, F_2 and F_3 act on a body that remains in equilibrium.

F_1 has a magnitude of 400 N. The angle between the directions of F_1 and F_2 is 150°, between F_1 and F_3 is 135° and between F_2 and F_3 is 75°.

Determine the magnitudes of F_2 and F_3, rounding your answers to the nearest whole number. (4 marks)

Solution



For equilibrium, $F_1 + F_2 + F_3 = 0$ - nose to tail vectors in triangle.

If magnitudes of $F_2 = x$ and $F_3 = y$ then $\frac{x}{\sin 45} = \frac{y}{\sin 30} = \frac{400}{\sin 105}$

Solving gives $x = F_2 = 293$ N and $y = F_3 = 207$ N.

Specific behaviours

✓ sketch diagram

✓ triangle or equation indicates force vectors must sum to zero
✓ solves for x
✓ solves for y

Question 2 **(7 marks)**

(a) A number is to be formed by randomly selecting three **different** digits from those in the number 93265. Determine how many different numbers

(i) start with an odd digit. (1 mark)

Solution
$3 \times 4 \times 3 = 36$ numbers
Specific behaviours
✓ calculates correct number

(ii) end with an even digit. (1 mark)

Solution
$2 \times 4 \times 3 = 24$ numbers
Specific behaviours
✓ calculates correct number

(iii) start with an odd digit or end in an even digit. (2 marks)

Solution
$3 \times 2 \times 3 = 18$ numbers start with an odd digit and end in an even digit $36 + 24 - 18 = 42$ numbers
Specific behaviours
✓ calculates set intersection ✓ calculates correct number for set union

(b) A computer user has forgotten their six character, case-sensitive password, but know that they always use a permutation of F, F, 1, 9, 9, and 9 - their initials and the year they were born. Determine how many passwords are possible if

(i) the F's must both be uppercase. (2 marks)

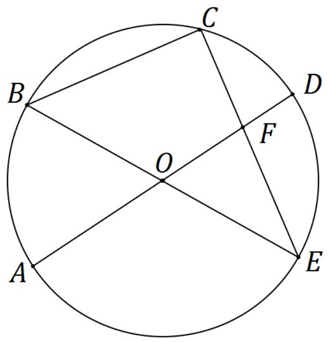
Solution
$\frac{6!}{2!\times 3!} = \frac{6\times 5\times 4}{2} = 60$ passwords
Specific behaviours
✓ shows correct method ✓ calculates correct number

(ii) either F can be lowercase or uppercase. (1 mark)

Solution
FF can be replaced with Ff, fF or ff, so $4 \times 60 = 240$ passwords
Specific behaviours
✓ calculates correct number

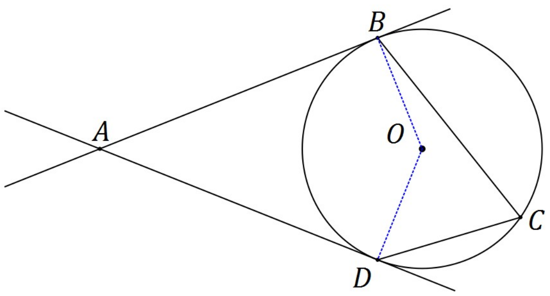
Question 3 **(8 marks)**

(a) Triangle BCE is such that B, C and E lie on a circle with centre O and radius 29 cm. Diameter AD and chord CE intersect at F, so that $DF = 8.5$ cm and $EF = 25.5$ cm. Determine the lengths OF, CF and BC. (5 marks)



Solution
$OF = 29 - 8.5 = 20.5$ cm $CF \times EF = DF \times AF \Rightarrow CF = 8.5 \times \frac{29+20.5}{25.5} = 16.5$ cm ΔBCE is right-angled $\Rightarrow BC = \sqrt{(29+29)^2 - (16.5+25.5)^2} = 40$ cm
Specific behaviours
✓ calculates OF ✓ uses intersecting chord theorem ✓ calculates CF ✓ uses angle in semicircle ✓ calculates BC

(b) In the diagram below, points B, C and D lie on a circle with centre O. The tangents to the circle at B and D intersect at point A. If $\angle BAD = x$, prove that $\angle BCD = 90° - \frac{x}{2}$. (3 marks)



Solution
In quadrilateral $ABOD$, $\angle ABO = \angle ADO = 90°$ (tangent-radius angle) $\angle BOD = 360 - 90 - 90 - x = 180 - x$ (angle sum in quadrilateral) $\angle BCD = \frac{1}{2} \times \angle BOD = \frac{1}{2} \times 180 - x = 90° - \frac{x}{2}$ (angle at centre twice that on circ.)
Specific behaviours
✓ adds radii to diagram noting right-angles ✓ determines $\angle BOD$ with reason ✓ determines $\angle BCD$ with reason

Question 4 **(9 marks)**

Transformation A is an anti-clockwise rotation about the origin of 90° and matrix $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

(a) Represent transformation A as a 2×2 matrix. (2 marks)

Solution
$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
Specific behaviours
✓ writes matrix for 90° rotation
✓ matrix is for anticlockwise rotation

(b) Describe the transformation represented by matrix B. (2 marks)

Solution
A dilation parallel to x-axis of scale factor 2 and dilation parallel to y-axis of scale factor 3.
Specific behaviours
✓ two dilations
✓ fully qualifies both dilations with directions and scale factors

(c) Determine the coordinates of the point $P(-15, -11)$ following transformation A and then transformation B. (2 marks)

Solution
$P' = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} -15 \\ -11 \end{bmatrix}$ $= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} 11 \\ -15 \end{bmatrix} = \begin{bmatrix} 22 \\ -45 \end{bmatrix}$
Specific behaviours
✓ calculates first image using A
✓ calculates P'

(d) Following transformation B and then transformation A, point Q is transformed to point $Q'(12, 7)$.

Determine the single matrix that will transform Q' back to Q and hence determine the coordinates of point Q. (3 marks)

Solution
$Q' = ABQ \Rightarrow Q = B^{-1}A^{-1}Q'$ $B^{-1}A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{3} & 0 \end{bmatrix}$ $Q = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{3} & 0 \end{bmatrix} \times \begin{bmatrix} 12 \\ 7 \end{bmatrix} = \begin{bmatrix} 3.5 \\ -4 \end{bmatrix}$
Specific behaviours
✓ determines inverse of B and inverse of A ✓ determines single matrix   ✓ determines coordinates of Q

Question 5 **(8 marks)**

(a) Consider the vectors $\mathbf{p} = (24, -143)$ and $\mathbf{q} = (20, -21)$. Determine

(i) the angle between the directions of vectors $\mathbf{p}$ and $\mathbf{q}$. (1 mark)

Solution
Using CAS, angle is 34.1°
Specific behaviours
✓ states correct angle.

(ii) two vectors that are perpendicular to $\mathbf{q}$ and have the same magnitude as $\mathbf{p}$. (3 marks)

Solution
Magnitude of required vectors are $\lvert\mathbf{p}\rvert = 145$ Unit vectors ⊥ to $\mathbf{q}$ are $\pm\frac{1}{29}(21, 20)$ Required vectors are $\pm\frac{145}{29} \times (21, 20) = (105, 100)$ and $(-105, -100)$
Specific behaviours
✓ calculates magnitudes of $\mathbf{p}$ and $\mathbf{q}$ ✓ determines at least one perpendicular vector   ✓ states both required vectors

(b) If $\overrightarrow{AB} = (3, 4)$ and $\overrightarrow{AC} = (-2, 1)$, determine

(i) the component of $\overrightarrow{AB}$ parallel to $\overrightarrow{AC}$. (2 marks)

Solution
Vector projection of $\overrightarrow{AB}$ on $\overrightarrow{AC}$: $\frac{(3, 4) \cdot (-2, 1)}{(-2, 1) \cdot (-2, 1)} \times (-2, 1) = (0.8, -0.4)$
Specific behaviours
✓ substitutes correctly into vector projection formula

(ii) the component of $\overrightarrow{AB}$ perpendicular to $\overrightarrow{AC}$. (2 marks)

Solution
Let component be $\mathbf{r}$. $\mathbf{r} + (0.8, -0.4) = \overrightarrow{AB}$ $\mathbf{r} = (2.2, 4.4)$
Specific behaviours
✓ shows use of vector addition   ✓ evaluates component

Question 6 **(7 marks)**

(a) The work done, in joules, by a force $\mathbf{F}$ Newtons in changing the displacement of an object $\mathbf{s}$ metres is given by the scalar product of $\mathbf{F}$ and $\mathbf{s}$. Calculate the work done when a force of 750 N moves an object a distance of 85 cm at an angle of 5° to the force.

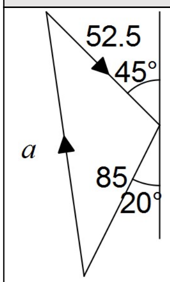
(2 marks)

Solution
$750 \times 0.85 \times \cos 5° = 635.1$ N
Specific behaviours
✓ substitutes correct values into scalar product ✓evaluates work done

(b) A drone flies with a constant velocity and height above level ground, over which a wind blows from the north west at 3.5 metres per second. After 15 seconds, the drone reaches a point 85 metres on a bearing of 020° from where it was launched. Determine the velocity of the drone, giving its magnitude to two decimal places and bearing to the nearest degree.

(5 marks)

Solution



Wind component is $15 \times 3.5 = 52.5$ m

$a^2 = 52.5^2 + 85^2 - 2(52.5)(85)\cos 115 \Rightarrow a = 117.2737$

Speed of drone is $117.2737 \div 15 = 7.82$ m/s (2dp).

$\frac{52.5}{\sin x} = \frac{106.4211}{\sin 115} \Rightarrow x = 24°$ and so bearing is $020 - 24 = 356°$.

Specific behaviours

✓ sketch displacement vector diagram

✓ uses cosine rule to determine drone displacement
✓ calculates drone speed
✓ uses sine rule to determine angle
✓ determines bearing

Question 7 **(8 marks)**

(a) A high school has 5 male and 9 female volunteers from which to choose a debating team of 5 students. Determine the number of different teams that can be formed if

(i) there are no special requirements. (1 mark)

Solution
$\binom{14}{5} = 2002$ teams
Specific behaviours
✓ evaluates correct number

(ii) there must be a captain and a vice-captain. (2 marks)

Solution
$\binom{14}{1} \times \binom{13}{1} \times \binom{12}{3} = 40040$ teams
Specific behaviours
✓ shows a suitable method
✓ evaluates correct number

(iii) there must be more females than males, but at least one male. (2 marks)

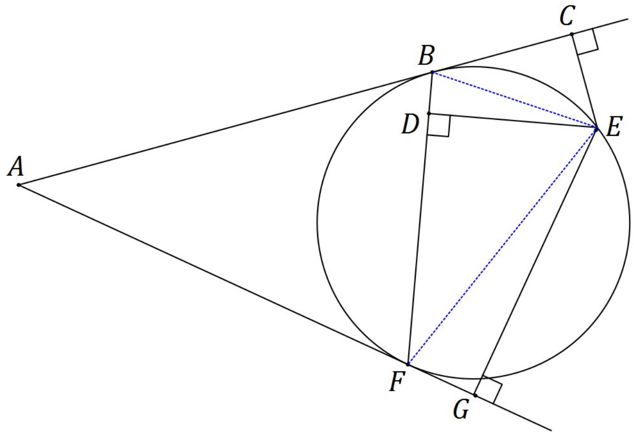
Solution
$\binom{5}{1} \times \binom{9}{4} + \binom{5}{2} \times \binom{9}{3} = 1470$ teams
Specific behaviours
✓ shows a suitable method

(b) Determine how many **different** numbers must be selected from the first 25 positive integers to be certain that at least one of them will be twice the other. (3 marks)

Solution
Partition integers into pigeonholes, where double included if possible: {1, 2}, {3, 6}, {4, 8}, {5, 10}, {7, 14}, {9, 18}, {11, 22}, {12, 24}, {13}, {15}, {16}, {17}, {19}, {20}, {21}, {23}, {25}. *NB Other partitions possible.* There are 17 partitions (pigeonholes) and so 18 numbers (pigeons) are required to ensure that at least one will be twice the other.
Specific behaviours
✓ partitions most integers
✓ systematically partitions all integers

Question 8 **(7 marks)**

In the diagram below, the tangents from point A touch the circle at B and F. Point E lies on the major arc BF and D lies on BF so that $DE \perp BF$. Points C and G lie on AB and AF extended respectively such that $EC \perp AC$ and $EG \perp AG$.



(a) Show that ΔBCE and ΔFDE are similar. (3 marks)

Solution
$\angle CBE = \angle DFE$ (alternate segment theorem)
$\angle BCE = \angle FDE$ (both right)
Hence $\Delta BCE \sim \Delta FDE$ (AA)
Specific behaviours
✓ shows one pair of angles equal with reason
✓ shows second pair of angles equal with reason
✓ makes conclusion that similar

(b) Show that $DE^2 = CE \times GE$. (4 marks)

Solution
$\angle GFE = \angle DBE$ (alternate segment theorem)
$\angle BDE = \angle FGE$ (both right)
Hence $\Delta BDE \sim \Delta FGE$ (AA)
So ratio of sides is $\frac{DE}{GE} = \frac{BE}{FE}$
But ratio of sides from (a) is $\frac{CE}{DE} = \frac{BE}{FE} \Rightarrow \frac{CE}{DE} = \frac{DE}{GE} \Rightarrow DE^2 = CE \times GE.$
Specific behaviours
✓ realises second pair of similar triangles required
✓ uses same reasoning from (a) to show $\Delta BDE \sim \Delta FGE$
✓ states ratio of corresponding sides for both pairs of triangles
✓ uses common ratio to obtain result